Matrices

Definition: An m x n matrix is an arrangement of mn objects (not necessarily distinct) in m rows and n columns in the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

We say that the matrix is of order m x n (m by n). The objects $a_{11}, a_{12}, \ldots, a_{mn}$, are called the elements of the matrix.

$A = [a_{ij}]$

$B = [b_{ij}]$

Each element of the matrix can be a real or a complex number or a function of one more variables or any other object. The element a; which is common to the $i\,th$ row and the $j\,th$ column is called its general element. The matrices are usually denoted by boldface uppercase letters , C, ... etc. When the order of the matrix is understood, we can simply write $A = [a_{ij}]$. If all elements of a matrix are real, it is called a real matrix, whereas if one or more elements of a are complex it is called a complex matrix.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Real

$B = \begin{bmatrix} i & 2+i \\ 3 & 4 \end{bmatrix}$

Types of Matrices

1. Row Vector: A matrix of order 1 x n that is, it has one row and n column is called row matrix or row vector. And it can be written as $[a_{11}, a_{12}, \ldots, a_{1n}]$ in which a_{1j} is the j th element.

$[1, 2, 3, 4]$ 4 1×4

$[a_{11}\ a_{12}\ a_{13}]$ 1st 2nd 3rd

$[1\ 2\ 3\ 5]$

a) 1

b) 4

Q. What is the order of row vector ?

1. Column vector: A matrix of order mx1, that is, it has m row and one column is called column vector or column matrix of order m and is written as $\begin{bmatrix} \\ a_{m1} \end{bmatrix}$

Q. What is the order of column vector ?

3. Rectangular matrix: A matrix A of order m x n, $m \neq n$ is called a rectangular matrix.

4. Square matrices: A matrix A of order m x n in which m = n, that is number of rows is equal to the number of columns is called a square matrix of order n.

. diagonal elements

. principal diagonal

. principal diagonal

. off-diagonal elements. a_{ij} $i \neq j$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

. Trace of the matrix.

a_{ij} $i = j$

$$A = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\text{Tra}(A) = 5 + 8 = 13$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \\ 6 & 7 & 2 \end{bmatrix}$$

a_{13} $1 \neq 3$

(a) 0 (c) 2
(b) 4 (d) 3

1. ~~Null matrix~~: A matrix A of order mxn in which all the elements are zero is called a null matrix or a zero matrix and is denoted by 0

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$ $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times2}$

$A^2 + 2A = 0$

$A_{2\times2}$ $A = [0]_{2\times2}$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

$]_{3\times2}$

$]_{1\times3}$

Q. What is order of the null matrix

6. Diagonal Matrix: A square matrix A in which all the off-diagonal elements a_{ij} $i \neq j$ are zero is called a diagonal matrix. For example

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\text{dig}(1, 2)$ $\text{dig}(1, 1, 1)$ $\text{dig}(1, 2, 3)$

7. Unit Matrix

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = B$

1. Equal matrix

$A = [a_{ij}]$ $B = [b_{ij}]$

1. Equal matrix

$A = [a_{ij}]$ $B = [b_{ij}]$

$A = B = ?$

$m \times n$ $m \times n$

$- b_{ij}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} x+2 & 2 \\ 3 & 4 \end{bmatrix}$ $x+2 = 3$, $x = 1$

$a_{11} =$

2. Sub Matrix

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$

$[1\ 2\ 3]$ yes No

$\begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$ yes No

1. Scalar Matrix

$A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}_{2\times 2} = [5]_{2\times 2}$ $B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Example 1. *Find the values of x, y, z and 'a' which satisfy the matrix equation.*

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix} \quad 2\times 2$$

$x+3=0$, $x=-3$ $z-1=3$, $z=4$ $4a-6=2a$, $2a=6$, $a=3$.

a. -3, -2, 4, -3
b. -2,-3,4,3 ✓
c. -3, -2, 4, 3

$2y+x=-7$
$2y-3=-7$
$\boxed{y=-2}$

Matrix Algebra

(i) Multiplication of a matrix by a scalar,
If a matrix is multiplied by a scalar quantity k, then each element is multiplied by k, i.e.

$A = [a_{ij}]$

$A = [a_{ij}]$

$K.A = [Ka_{ij}]$

$KA = \begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix}$

https://www.geogebra.org/m/jaJwgaar

$A = [a_{ij}]_{m\times n}, B = [b_{ij}]_{m\times n}$

$A + B = [a_{ij} + b_{ij}]$

(ii) Addition/subtraction of two matrices,

https://www.geogebra.org/m/Thmhe5T

$$A = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 3 & -6 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4+1 & 2+0 & 5+2 \\ 1+3 & 3+1 & -6+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 4 & 4 & -2 \end{bmatrix}$$

Note: Only matrices of the same order can be added or subtracted.

(i) Commutative Law: A + B = B + A.

(ii) Associative law: A + (B + C) = (A + B) + C.

$\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & x+y+6 \\ z+w-1 & 2w+3 \end{bmatrix}$

Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

$3w = 2w + 3$

$\boxed{w = 3}$

Find x, y, z and w.

a. 2,4,1,3
b. 2,1,3,4
c. 4,2,3,4
d. 1,3,2,4

3×3

(iii) Multiplication of two matrices.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$$

3×2

$AB =$

$\begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$

$0\cdot 1 + 1(-1) + 2(2)$

$0 + (-1) + 4$

$0(-2) + 1(0) + 2(-1$

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \times \begin{array}{cc} C_1 & C_2 \\ \end{array} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} R_1\,C_1 & R_1\,C_2 \\ R_2\,C_1 & R_2\,C_2 \\ R_3\,C_1 & R_3\,C_2 \end{bmatrix}$$

https://www.geogebra.org/m/Thnmhe5T

PROPERTIES OF MATRIX MULTIPLICATION

1. Multiplication of matrices is not commutative: $AB \neq BA$
2. Matrix multiplication is associative, if conformability is assured. A (BC) = (AB) C
3. Matrix multiplication is distributive with respect to addition. A (B + C) = AB + AC
4. Multiplication of matrix A by unit matrix. AI = IA = A
5. Multiplicative inverse of a matrix exists if $|A| \neq 0$.
$A\,.A^{-1} = A^{-1}.A = I$
6. If A is a square then $A \times A = A^2, A \times A \times A = A^3$.
7. $A^0 = I$
8. $I^n = I$, where n is positive integer.

Some special Matrices

Transpose of the matrix:

If in a given matrix A, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix A and is denoted by A' or A^T

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

Then A' or $A^T = ?$

$$A^T = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

Symmetric Matrix:

A square matrix will be called symmetric, if for all values of i and j, $a_{ij} = a_{ji}$ i.e., $A = A^T$

$$\begin{bmatrix} a & h & g \\ \end{bmatrix}$$

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

Skew symmetric matrix: A square matrix is called skew symmetric matrix, if

(1) $a_{ij} = -a_{ji}$ for all values of i and j, or $A^T = -A$

(2) All diagonal elements are zero,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

Triangular matrix: (Echelon form) A square matrix, all of whose elements below the leading diagonal are zero, is called an upper triangular matrix. A square matrix, all of whose elements above the leading diagonal are zero, is called a lower triangular matrix

$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}_{2\times 2}$

e.g.,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

Upper triangular matrix

$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}_{2\times 2}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Lower triangular matrix

Orthogonal Matrix: A square matrix A is called an orthogonal matrix if the product of the matrix A and the transpose matrix A′ is an identity matrix

$AA^T = I$

e.g. $A.A^T = I$

EX... ???? Ex $A = [1 \quad 0$

Note: if | A | = 1, matrix A is proper.

~~Conjugate matrix~~:

$c = a+ib$

$\bar{c} = a-ib$

$$\bar{A} = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

Hermitian Matrix:

A square matrix $A = (a_{ij})$ is called Her~~mitian~~ matrix, if every i-jth element of A is equal to conjugate complex j-ith element of A. That means

$$a_{ij} = \bar{a}_{ji}$$

$$A = (\bar{A})^T$$

Ex...???

$$A = \begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} 1 & -i \\ i & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{1} & 2+3i & 3+i \\ 2-3i & \cancel{2} & 1-2i \\ 3-i & 1+2i & \cancel{5} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2+i & 3i \\ 2-i & 5 & 2+2i \\ -3i & 2-2i & 6 \end{bmatrix}$$

Skew-Hermitian Matrix:

A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every i-jth element of A is equal to negative conjugate complex of j-ith element of A. That means

$$a_{ij} = -\bar{a}_{ji}$$

$$-A = (\bar{A})^T$$

conjugate complex of j-ith element of A. That means

$a_{ij} = -\bar{a}_{ji}$

$-A = (\bar{A})'$

$$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

Note: All the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary

$-A = (\bar{A})^T$ — Step-1

$-[a_{ij}] = [\bar{a}_{ij}]^T$

$-[a_{ij}] = [\bar{a}_{ji}]$

$-a_{ii} = \bar{a}_{ii}$

$-(a+ib) = (\overline{a+ib})$

$-(a)-ib = a-ib$

$-2a \; -ib+ib = 0$

$\boxed{a=0}$

$a_{ii} = a+ib$

$a_{ii} = a+ib$

$= ib$

$a_{ii} = a+ib$

$= ib$

$-a_{ii} = \bar{a}_{ii}$

$-(a+ib) = (\overline{a+ib})$

$-a-ib = a-ib$

$-a-a-ib+ib = 0$

$-2a=0 \Rightarrow \boxed{a=0}$

$A^\theta = (\bar{A})^T = A^\theta$

Matrix A^θ: Transpose of the conjugate of a matrix A is denoted by A^θ.

Note: Necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^\theta$ i.e. conjugate transpose of A

$$\Rightarrow \qquad A = (\bar{A})^T$$

Unitary Matrix: A square matrix A is said to be unitary if $A^\theta A = I$

$A \cdot A^\theta = I$

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \quad \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A^\theta A = A \cdot A^\theta$

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \quad \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & \\ & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A^\theta A = A \cdot A^\theta$

$= I$

$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Idempotent Matrix: A matrix, such that $A^2 = A$ is called Idempotent Matrix.

$A^2 = A$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = [\quad] A$$

Periodic Matrix: A matrix A will be called a Periodic Matrix, if $A^{k+1} = A$ where k is a +ve integer. If k is the least + ve integer, for which $A^{k+1} = A$, then k is said to be the period of A. If we choose k = 1, we get $A^2 = A$ and we call it to be idempotent matrix. For ex. $A^2 = A$

$k = (a)\ 1$
$(b)\ 2$
$(c)\ 3$

$1+6-12$

$-2-4$

$$\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -5 & -6 & -6 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & -2 & -6 \\ & & \end{bmatrix}$$

$-5+18-12$

Nilpotent Matrix: A matrix will be called a Nilpotent matrix, if $A^k = 0$ (null matrix) where k is a +ve integer ; if however k is the least +ve integer for which $A^k = 0$, then k is the index of the nilpotent matrix.

$A^k = 0$ index

$K = 2$
index $= 2$

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}, \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

Involuntary Matrix: A matrix A will be called an Involuntary matrix, if $A^2 = I$ (unit matrix). Since $I^2 = I$ always

So we can say: Unit matrix is involuntary.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$A = \begin{bmatrix} 4 & -1 \\ 15 & -4 \end{bmatrix} \quad \begin{bmatrix} 4 & -1 \\ 15 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \quad 3\times 3$$

Singular Matrix: If the determinant of the matrix is zero, then the matrix is known as singular matrix

$|A|$ or $\det(A) = 0$ Singular

$|A| \neq 0$ Non Singular

Determinant

Determinant of second order

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad \text{or} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} 4 & 6 \\ 2 & 5 \end{vmatrix} = 20 - 12 = 8 \qquad \begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix} = 8 - 15 = -7$$

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$\begin{vmatrix} -3 & 7 \\ 2 & 4 \end{vmatrix} = -12 - 14 = -26 \qquad \begin{vmatrix} 5 & -2 \\ 4 & 3 \end{vmatrix} = 15 + 8 = 23$$

$\cos^2\theta + \sin^2\theta$
$= 1$

MINOR: The minor of an element is defined as a determinant obtained by deleting the row and column containing the element.

For ex.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2 c_3 - b_3 c_2$$

Thus the minors a_1, b_1 and c_1 are respectively.

$$\boxed{\underset{a_1}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}, \quad \underset{b_1}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} \text{ and } \underset{c_1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}}$$

So the determinant can be found as follows

$$\det(A) = \boxed{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underset{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{(\text{minor of } a_1)} - b_1 \underset{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{(\text{minor of } b_1)} + c_1 \underset{\begin{vmatrix} a_2 & b_2 \\ a_3 & c_3 \end{vmatrix}}{(\text{minor of } c_1)}.}$$

COFACTOR

COFACTOR:

$$\text{Cofactor} = (-1)^{r+c} * \text{Minor}$$

where r is the number of rows of the element and c is the number of columns of the element. The cofactor of any element of jth row and ith column is $(-1)^{i+j} *$minor

Ex. Find the Minors and cofactors of first row

a_{11} $A = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix}$

a_{12}

Minor (2) $= \begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} = 7$

Cofa (2) $= (-1)^{1+1} \cdot 7 = 7$

Min (3) $= \begin{vmatrix} 4 & 0 \\ 6 & 7 \end{vmatrix} = 28$

Cof (3) $= (-1)^{1+2} \cdot 28 = -28$

$\det(A) = 2(7) - 3(28) + 5(8-6)$ $|A|$

Ex. Find the determinant of $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix}$ 4×4 3×3

$0\begin{vmatrix} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix}$

$0 - 1(-3(1-2)) + 2(1(-6)) - 3(1(-2) + 2(4)) = $

PROPERTIES OF DETERMINANTS

1. The value of a determinant remains unaltered, if the rows are interchanged into columns

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21} \cdot a_{12}$$

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

2. If two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant changes.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

$$a_{11} \cdot a_{22} - a_{21}a_{12} \qquad = a_{21} \cdot a_{12} - a_{11}a_{12}$$

$$-(a_{11}a_{22} - a_{21} \cdot a_{12})$$

3. If two rows (or columns) of a determinant are identical, the value of the determinant is zero.

$$\begin{vmatrix} a & a \\ b & b \end{vmatrix} = 0 \qquad \begin{vmatrix} a & ka \\ b & kb \end{vmatrix} = kba - kba = 0$$

4. If the elements of any row (or column) of a determinant be each multiplied by the same number, the determinant is multiplied by that number.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc)$$

5. The value of the determinant remains unaltered if to the elements of one row (or column) be added any constant multiple of the corresponding elements of any other row (or column) respectively.

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 5 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 1 \\ 8 & 1 & 4 \\ 12 & 6 & 7 \end{vmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \begin{vmatrix} a+kb & b \\ c+kd & d \end{vmatrix}$$

$$ad - bc \;=\; ad + \cancel{kbd} - bc - \cancel{kbd} = ad - bc$$

18 67

6. If each element of a row (or column) of a determinant consists of the algebraic sum of n terms, the determinant can be expressed as the sum of n determinants.

$$\begin{vmatrix} a+b+c & d \\ e+f+J & k \end{vmatrix} = \begin{vmatrix} a & d \\ e & k \end{vmatrix} + \begin{vmatrix} b & d \\ f & k \end{vmatrix} + \begin{vmatrix} c & d \\ J & k \end{vmatrix}$$

Same Ans

Application of determinant: To find the area of the triangle

Ex. Using determinants, find the area of the triangle with vertices (– 3, 5), (3, – 6) and (7, 2).

$$\frac{1}{2}\begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 7 & 2 & 1 \end{vmatrix} = \_\_\_$$

Ex. Using determinants, show that the points (11, 7), (5, 5) and (– 1, 3) are collinear

(x_1, y_1)
(x_2, y_2)
(x_3, y_3)

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\frac{1}{2}\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ -1 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left(11(2) - 7(6) + 1(20)\right)$$
$$= \frac{1}{2}(22 - 42 + 20) = 0$$

Singular Matrix: If the determinant of the matrix is zero, then the matrix is known as singular matrix

$|A| = 0$
A is Singular
$|A| \neq 0$ non Sin

RANK OF A MATRIX

r

$|\text{Min}| \neq 0$

The rank of a matrix is said to be r if ... $= 0$

RANK OF A MATRIX

$|M_n| \neq 0$

$|\ |_{r+1} = 0$

The rank of a matrix is said to be r if

(a) It has at least one non-zero minor of order r.

(b) Every minor of A of order higher than r is zero.

2×2

Find the rank of the matrix $\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

$\begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = 9 - 15 = -6$ $\quad r = 2$, $\rho(A) = 2$

Find the rank of the matrix $\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

$[-5]$

$\begin{vmatrix} -5 & -7 \\ 5 & 7 \end{vmatrix} = -35 + 35 = 0$ $\quad 2\times 2 = 0$, 1×1

$r = 1$, $\rho(A) = 1$

Find the rank of the matrix $\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$

$3\times 3 = 0$, $\neq 0$

$0(\ ) + 1(10+6) + 5(2-4) = 16 - 10 = 6 \neq 0$

$r = 3$

Find the rank of the matrix $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$

3×3, 2×2, 1×1

$5(16-16) - 3(8-8) + 0(\ ) = 0$ $\quad 3\times 3 = 0$

2×2

$\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7 \neq 0$

Find the Rank of using echelon form

(1) $R_i \leftrightarrow R_j$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$$

$R_2 - 2R_1$, $R_3 - 3R_1$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$

$R_2 - 2R_1$: $4 - 2(3)$, $4 - 6 = -2$

$3 - 3$, $5 - 3(2)$, $7 - 3(3)$

(I) non zero

$R_3 - R_2$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ (1), (2)

$r = 2$

$\rho(A) = 2$

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$	(2) Rank = 2
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ (1), (2) $\rho(A) = 2$	$R_3 \to R_3 - R_2$
The above matrix is in echelon form	

The above matrix is in echelon form | $R_3 \to R_3 - R_2$

$-x_2 - x_2 2 =$

Find the rank of the matrix A= $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$ 1 2 3	$R_3 \to R_3 + 5R_2$

Rank = 3

0 0 1

0 1 2

Note: (i) Non-zero row is that row in which all the elements are not zero.

Rank

How many Rows are there

(ii) The rank of the product matrix AB of two matrices A and B is less than the rank of either of the matrices A and B.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad 4\times4$$

$R_1 \leftrightarrow R_2$ 4×4

$|A|$

$|A_1|$

$|A_2|$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

I) $R_i \leftrightarrow R_j$

II) $R_i \to k \cdot R_i$

III) $R_i \to R_i + kR_j$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$-2+12$

$3+6$

$-7+24$

$R_4 \to R_4 - \frac{9}{5}R_2$

$R_3 \to R_3 - \frac{4}{5}R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \end{bmatrix}$$

$9 - 4$

$R_4 \to r_4$

$R_2 \to R_2^{-1}$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & \frac{33}{5} & \\ 0 & & & \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{5}R_2$

$9 - \frac{4}{5}(3)$

$\frac{45-12}{5} = \frac{33}{5}$

$10 - \frac{4}{5}7$

$\frac{50-28}{5} =$

Linearly dependence and independence of vectors:

$X_1 = [a_{11}\ a_{12}\ a_{13}]^T$

$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}$

row matrix — Col Matr

$1 \times m$ row matrix vector

$m \times 1$

$1 \times m$

$m \times 1$

Vectors (matrices) $X_1, X_2, \ldots . X_n$ are said to be dependent if

(1) all the vectors (row or column matrices) are of the same order.

$\times\ [1, 2, 3]\ (1, 2)$

(2) n scalars $C_1, C_2, \ldots C_n$ (not all zero) exist such that

$X_1 X_2 - X_n$

$X_1 X_2$

$X_1 X_2 X_3$

$$C_1X_1 + C_2X_2 + \cdots + C_nX_n = 0$$

$C_1 = C_2 = C_3 = \ldots = C_n = 0$ L.I

but if at least one $C_i \neq 0$ L.D

L.D ~~LI~~

Otherwise they are linearly independent.

Find whether or not the following set of vectors are linearly dependent or independent:

Ex 1

(i) (1, -2), (2, 1)

$X_1 \quad X_2$

$C_1X_1 + C_2X_2 = 0$

$C_1(1,-2) + C_2(2,1) = 0$

$C_1 + 2C_2 = 0$ — (1)

$-2C_1 + C_2 = 0$ — (2)

$-(1)\times 2 + (2)$

$-(1)\times 2 + (2)$

$$-2c_1 + c_2 = 0 \quad -(2)$$

$$\cancel{2c_1} + 4c_2 = 0$$
$$-\cancel{2c_1} + c_2 = 0$$
$$5c_2 = 0 \Rightarrow \boxed{c_2 = 0}$$

$$\boxed{c_1 = 0}$$

L.I

(ii) (1, -2), (2, 1), (3, 2)

$x_1 \quad x_2 \quad x_3$

$c_1 \quad c_2 \quad c_3$

(c_1, c_2, c_3)

$$c_1x_1 + c_2x_2 + c_3x_3 = 0$$

$$c_1(1,-2) + c_2(2,1) + c_3(3,2) = 0$$

$$c_1 + 2c_2 + 3c_3 = 0 \quad -(1)$$
$$-2c_1 + c_2 + 2c_3 = 0 \quad -(2)$$

$-(1)\times 2 + (2)$

$c_3 =$

$$\cancel{2c_1} + 4c_2 + 6c_3 = 0$$
$$-\cancel{2c_1} + c_2 + 2c_3 = 0$$

$$5c_2 + 8c_3 = 0$$

L.D.

$c_2 =$

$$c_2 = -\frac{8}{5}c_3$$

$(-\infty, \infty)$

c_3 c_2 c_1

$c_3 = 1$	$c_3 = -1$	$c_3 = 2$
$c_2 = -\frac{8}{5}$	$c_2 = \frac{8}{5}$	$c_2 = \frac{16}{5}$
$c_1 =$	$c_1 =$	c_1

$$c_2 = -\frac{8}{5}c_3$$

$$\boxed{c_3 = 0 \quad c_2 = 0 \quad c_1 = 0}$$

(iii)(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1).

$x_1 \quad x_2 \quad x_3 \quad x_4$

Defn

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$$

$$c_1 = 0$$
$$c_1 + c_2 = 0 \qquad c_2 = 0$$
$$c_1 + c_2 + c_3 = 0 \qquad c_3 = 0$$
$$c_1 + c_2 + c_3 + c_4 = 0 \qquad c_4 = 0$$

L.I.

Method 2

LINEARLY DEPENDENCE AND INDEPENDENCE OF VECTORS BY DETERMINANT: -

If the determinant of the these $X_1, X_2, X_3, \dots, X_n$ is zero then they are dependent otherwise independent.

$|A| = 0$ L.D

$|A| \neq 0$ L.I

$|A| = |A^T|$

Ex. (i) (1, −2), (2, 1)

$|A| = 0$ L.D

$|A| \neq 0$ L.I

$\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1+4 = 5 \neq 0$ L.I

$\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}$

$(3,3,2) = (1,2,1) + (2,1,1)$

(ii)(1,2,1),(2,1,1)(3,3,2)

L.D

$y = f(x)$

y depen

x indep

$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{vmatrix}$

$= 1(2-3) - 2(4-3) + 1(6-3)$

$= -1 - 2(1) + 3 = 0$ L.D.

(iii)$(3,0,1)^T(1,2,1)^T(2,-2,0)^T$

$|A| = |A^T|$

$\neq 0$

$\begin{vmatrix} 3 & 1 & 2 \\ 0 & 2 & -2 \\ 1 & 1 & 0 \end{vmatrix}$

$3(0+2) - 1(+2) + 2(-2)$

$6 - 2 - 4 = 0$ L.D

(iv) (1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1).

$\rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1\left(1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\right)$

0 0 0 1

Defn

$= 1.1.1 = 1 \neq 0$ L.I

$|A| = 0$ L.D

$|A| \neq 0$ L.I

LINEARLY DEPENDENCE AND INDEPENDENCE OF VECTORS BY RANK METHOD:

$X_1 \; X_2 \; X_3$

1. If the rank of the matrix of the given vectors is equal to number of vectors, then the vectors are linearly independent.

3

2

Rank A = No of vectors given Stm.

2. If the rank of the matrix of the given vectors is less than the number of vectors, then the vectors are linearly dependent.

Rank (A) = No of vectors L.I

< No of V L.D

Rank < No of vectors

L.D

Ex. The following vectors are linearly independent or not

$A = [X_1 \; X_2 \; X_3]$

$X_1 = (2,2,1)^T, X_2 = (1,3,1)^T, X_3 = (1,2,2)^T$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

$|A| = 2(6-2) - 1(4-2) + 1(2-3)$

$= 8 - 2 - 1 = 5 \neq 0$ Rank = 3

Rank = No of v.

L.I

Ex. Show using a matrix that the set of vectors

X = [1, 2,– 3, 4], Y = [3, – 1, 2, 1] , Z = [1, –5, 8, –7] is linearly dependent.

(2)

$\begin{vmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \end{vmatrix}$

X = [1, 2, - 3, 4], Y = [3, - 1, 2, 1] , Z = [1, -5, 8, -7] is linearly dependent.

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 1 \\ 1 & -5 & 8 & -7 \end{bmatrix}$$

3×4

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 1 & -5 & 8 \end{vmatrix}$$

3×3 ≠ 0

Rank 3

A 6×2

(a) 6 X
(b) 5 X
(c) 4 X
(d) 2 ✓

3×3

Rank (A) ≤ min (3, 4)

$\boxed{\text{Rank } (A) \leq \min(m, n)}$

L.I. L.D

vec. [(x, y, z), (2x, 3y, z)]

rk 2

$$\frac{y}{f_1} = e^8, \quad \frac{y^2 z}{x} = e^4, \quad \frac{x^3 y}{z^4} = 1$$

f_2 f_3

$$\left| \begin{matrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{matrix} \right| = 0 \text{ L.D}$$

≠ 0 L.I

$c_1 X_1 + c_2 X_2 + \ldots + c_n X_n = 0$

→ $c_i = 0 \; \forall i$ L.I

→ at least one $c_i \neq 0$ L.D

ADJOINT OF A SQUARE MATRIX

Let the determinant of the square matrix A be | A |.

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}_{3\times 3}$, Than $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

$\begin{bmatrix} M_{11} & M_{12} & M_{13} \end{bmatrix}^T =$

The matrix formed by the co-factors of the elements in

$|A|$ is $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$.

$a_{12} = (-1)^{1+2} M_{1n}$

where $A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2$, $\quad A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -b_1c_3 + b_3c_1$

$A_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1$, $\quad B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -a_2c_3 + a_3c_2$

$B_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$, $\quad B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -a_1c_2 + a_2c_1$

$C_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$, $\quad C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -a_1b_3 + a_3b_1$

$C_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

Then the transpose of the matrix of co-factors

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$ is called the adjoint of the matrix A and is written as adj A.

Ex. Find the adjoint of the matrix $\begin{array}{ccc} + & - & + \\ \end{array}$ $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

$1+3 \cdot 4$

$= 4-6 = -2$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\text{adj} A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

$\text{adj} A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

PROPERTY OF ADJOINT MATRIX

$(\text{adj} A)^T = A^T \quad (|A| \cdot I_n)^T$

$\text{adj} A^T = \frac{1}{|A|} \cdot A$

(1) $A.adj(A) = adj(A).A = |A|.I_n$ where, A is a square matrix, I is an identity matrix of same order.

$A \cdot \text{adj} A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2 I_n$

$|A| \neq 0 \quad A^T$

(2) if A is invertible square matrix

$|adj(A)| = |A|^{n-1}$

$A \quad A^{-1}$ exist

invertible

$(\text{adj} A)^T$

$$|adj(A)| = |A|^{n-1}$$

(3) if A is invertible square matrix

$$adj(adj(A)) = |A|^{n-2}.A$$

if $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is adjoint of 3×3 matrix with $|A| = 4$ then the value of α is

a. 2
b. 11
c. 13
d. 1/3

$1(12-12)-\alpha(-2)+3(-2)=4^2$

$0+2\alpha-6=16$

$2\alpha=22$

$\alpha=11$

(4) $adj(AB) = adj(B).adj(A)$

(5) $(adj(A))^T = adj(A^T)$

(6) $adj(kA) = k^{n-1}adj(A)$

INVERSE OF A MATRIX

If A and B are two square matrices of the same order, such that AB = BA = I (I = unit matrix) then B is called the inverse of A i.e. B = A^{-1} and A is the inverse of B.

To find the inverse of the Matrix A we use

$$A^{-1} = \frac{1}{|A|}(Adj(A)), \quad if\ |A| \neq 0$$

Properties of inverse of the matrix

1. Inverse of the matrix is unique

Properties of inverse of the matrix

1. Inverse of the matrix is unique
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is an invertible square matrix; Then $(A)^T$ is also invertible and $(A^T)^{-1} = (A^{-1})^T$
4. The inverse of an invertible symmetric matrix is a symmetric matrix.

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

A is invertible ⟹ A^{-1} exists ⟹ $|A| \neq 0$

$AB = BA = I$
$AB' = B'A = I$
$B = B'$

$|A| = |A^T|$

$$\begin{bmatrix} a & f & g \\ f & b & h \\ g & h & c \end{bmatrix}^{-1} = \begin{bmatrix} a' & f' & g' \\ f' & b' & h' \\ g' & h' & c' \end{bmatrix}$$

5. $|A^{-1}| = |A|^{-1}$ i.e. $|A^{-1}| = \dfrac{1}{|A|}$

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad |A| = 6$ $\qquad |A^{-1}| = |A|^{-1}$

$A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}' = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ $\qquad |A^{-1}| = \frac{1}{6}$ $\quad (6)^{-1} = \frac{1}{6}$

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad |A| = 8$

$|A^{-1}| = \begin{vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{8}$ $\quad |A|^{-1}$

Solution of $n \times n$ linear system of equation

Consider the system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \quad (1) \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \quad (2) \\ &\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n. \quad (3) \end{aligned}$$

$2x + 3y = 5$
$3x + 2y = 8$
$4x + 7y = 8$

In matrix form we can write this system as $\boxed{Ax = b}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},\quad \mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},\quad \mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Note:

1. A is the coefficient matrix, b the right hand side, and x is the solution vector.
2. If b not equal to zero system is called non-homogeneous.
3. If b is zero its call homogeneous.

$x = \begin{bmatrix} x_1 \neq 0 \\ x_2 \neq 0 \\ x_3 \neq 0 \\ x_4 \neq 0 \end{bmatrix}$

$Ax = b \neq 0$

$b = 0$

4. The system of equations is called consistent if it has at least one solution.

   otherwise the system is inconsistent.

$Ax = b$: 1, ∞ — Consistent; No — inconst

$xy=$

Homogeneous system of equations:

Consider the homogeneous system of equations $Ax = 0$ — 1, ∞

Zero

Trivial solution $x = 0$ is always a solution of this system.

If A is non singular, then $x = A^{-1}0 = 0$ is the solution.

$2x+y=0$

$3x+4y=0$

$x=0$

$y=0$

Thus $Ax = 0$ is the always consistent .

We conclude that non-trivial solution for $Ax = 0$ exist if and only if A is singular, in this case this system has infinite solutions.

$Ax = 0$ if $|A| \neq 0$, A^{-1}

$|A| = 0$ ∞ many Sol

$A^{-1}.Ax = A^{-1}.0$

$\boxed{x = 0}$ zero

$Ax = 0$:
- 1: $|A| \neq 0$ — $x=0, y=0, z=0$
- ∞: $|A| = 0$ — non zero Sol

zero ✓

no zero

$|A| = 1 \neq 0$ $x, y, z = 0$

Ex. Solve the system of the equation using

$x - y + z = 0, \quad 2x + y - 3z = 0, \quad x + y + z = 0$ $x = y = 2$

$AX = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x = 0$, $y = 0$, $z = 0$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\boxed{Ax = 0}$

$x - y + z = 0$

$0 - 0 + 0 = 0$

L.H.S = R.H.S

$|A| = 0$

Ex. If the system of the equations $x - ky - z = 0, kx - y - z = 0, x + y - z = 0$ has non-zero solution then values of K are

a. -1,2

b. 0,1

c. 1,1

d. -1,1

$|A| = 0$

$$\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$1(1+1) + k(-k+1) - 1(k+1)$

$2 - k^2 + \cancel{k} - \cancel{k} - 1$

$-k^2 + 1 = 0$

$k^2 = 1$

$k = \pm 1$

Ex. If the system of the equations $kx + y + z = 0, -x + ky + z = 0, -x - y + kz = 0$ has non-zero solution then value of K is

a. 0

b. 1

c. -1

d. 2

$$\begin{vmatrix} k & 1 & 1 \\ -1 & k & 1 \\ -1 & -1 & k \end{vmatrix} = 0$$

$k(k^2+1) - 1(-k+1) + 1(1+k) = 0$

$k(k^2+1) + k - 1 + 1 + k$

$k(k^2+1+2) = 0$

$k = 0$

$k = \pm\sqrt{3}i$

Solution of Non-homogeneous system of equations

$Ax = b$

The non-homogeneous system of equations $Ax = b$ can be solved by the following methods

(i) Matrix method

(ii) Cramer's Rule

(i) Matrix method:

Let A be non-singular, then pre-multiplying $Ax = b$ by A^{-1}, we obtain

$x = A^{-1}b$

$|A| \neq 0$, $\exists A^{-1}$

$$AX = b$$
$$A^{-1}AX = A^{-1}b$$
$$IX = A^{-1}b$$

$$X = \boxed{A^{-1}b}$$

Ex. Solve the system of the equation using matrix method $x - y + z = 4, \quad 2x + y - 3z = 0, \quad x + y + z = 2$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$|A| = 1(1+3) + 1(2+3) + 1(1) = 4 + 5 + 1 = 10$$

$$\operatorname{adj} A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \qquad A^{-1} = \frac{1}{10} \operatorname{adj} A$$

$$X = A^{-1}b = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Ex. Solve the system of the equation using matrix method $-x + y + 2z = 2, \quad 3x - y + z = 3, \quad -x + 3y + 4z = 6$

Ex. Solve the system of the equation using matrix method 2x-z=1, 5x+y=7, y+3z=5

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix}$$

$$|A| = 1$$

$$\operatorname{adj} A = \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \end{bmatrix}^T = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \end{bmatrix}$$

$3 - 7 + 5$

$$\text{adj}A = \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$3-7+5$

$-15+42-25$

$$X = A^{-1} \cdot b = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$Ax = b$ $\quad A^{-1}$

$x = A^{-1}b \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(ii) Cramer's Rule:

Let A be a non-singular matrix then by Cramer's rule solution of Ax=b is given by

x_i $\quad x_i = \frac{|A_i|}{|A|} \qquad i = 1,2,3,\dots,n$

$x_1 = x$, $x_2 = y$, $x_3 = z$

Where $|A_i|$ is the determinant of the matrix $|A_i|$ obtained by replacing the ith column of A by the right hand side column vector b.

x, y, z $\quad |A|$

$x_1 = x = \frac{|A_1|}{|A|} \quad x_2 = y = \frac{|A_2|}{|A|} \quad z = \frac{|A_3|}{|A|} \quad t = \frac{|A_4|}{|A|}$

Ex. Solve the system of the equation using

$x - y + z = 4, \qquad 2x + y - 3z = 0, \qquad x + y + z = 2$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$1(1+3) - 1(-5) + 1(1)$

$n \times n$

$$A_1 = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$\boxed{\begin{matrix} 2x+3y+z = 1 \\ x+y+z = 5 \end{matrix}}$ ✗

$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$ $\quad |A|$

$$A_3 = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$\lvert A\rvert$	$\lvert A_1\rvert$	$\lvert A_2\rvert$	$\lvert A_3\rvert$
10	20	-10	10

$x = \frac{|A_1|}{|A|} = \frac{20}{10} = 2 \qquad y = \frac{|A_2|}{|A|} = \frac{-10}{10} = -1$

$z = \frac{|A_3|}{|A|} = 1$

$x = 2 \quad y = -1 \quad z = 1$

$Ax = b$ $\quad x = 0$

1, No, ∞

Note: We have the following cases in this method

Case 1: when $|A| \neq 0$, the system is consistent and the unique

Note: We have the following cases in this method

$|A| \neq 0$

Case 1: when $|A| \neq 0$, the system is consistent and the unique solution is obtained by using the above method.

$|A_1|\ |A_2|$

Case 2: When $|A| = 0$, and one or more of $|A_i|, i = 1,2,3,\ldots,n$ are not zero then the system of the equations has no solution that is the system is inconsistent.

$|A| = 0$ at least $|A_i| \neq 0$

no sol

Case 3: When $|A| = 0$, and all $|A_i| = 0$, $\ i = 1,2,3,\ldots,n,$ then the system of equations is consistent and has infinite number of solutions . The system of equations has at least a one-parameter family of solutions.

Ex. Solve the system of the equation using

$4x + 9y + 3z = 6,\ 2x + 3y + z = 2,\ 2x + 6y + 2z = 7$

$$0 = |A| = \begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$$

$$0 = |A_1| = \begin{bmatrix} 6 & 9 & 3 \\ 2 & 3 & 1 \\ 7 & 6 & 2 \end{bmatrix} \qquad 0 \neq |A_2| = \begin{bmatrix} 4 & 6 & 3 \\ 2 & 2 & 1 \\ 2 & 7 & 2 \end{bmatrix}$$

$$|A_3| = \begin{bmatrix} 4 & 9 & 6 \\ 2 & 3 & 2 \\ 2 & 6 & 7 \end{bmatrix}$$

Ex. Solve the system of the equation using

$x - y + 3z = 3, \qquad 2x + 3y + z = 2, \qquad 3x + 2y + 4z = 5$

$$|A| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 0 \qquad |A_1| = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 0$$

$$|A_2| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 0 \qquad |A_3| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0$$

$$\begin{cases} x - y + 3z = 3 & \text{(1)} \\ 2x + 3y + z = 2 & \text{(2)} \\ 3x + 2y + 4z = 5 & \text{(3)} \end{cases}$$

(2) − (1) × 2

$5y - 5z = -4$

(3) − 3 × (1)

$5y - 5z = -4$

$5y - 5z = -4$

$5y - 5z = -4$

$-②-3\times-①$

$y = -\frac{4}{5} + z$

a) 1
b) No
c) ∞

$z = t \qquad y = -\frac{4}{5} + t \qquad x = f(t)$

Ex. The system of linear equations $x + y + z = 2, 2x + 3y + 2z = 5, 2x + 3y + (a^2 - 1)z = a + 1$

a. Is inconsistent for $a = 4$

b. Has unique solution for $a = \sqrt{3}$

c. Has infinite solution for $a = 4$

d. Inconsistent for $a = \sqrt{3}$

$|A| \neq 0$ — 1

$|A| = 0$ — No, ∞

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & a^2-1 \end{vmatrix}$$

$1(3(a^2-1)-6) - 1(2(a^2-1)-4) + 1(0)$

$a^2 - 3 = 0$

$16 - 3 = 13 \neq 0$ ①

Ex.

If the system of linear equation x- 4y+7z = g, 3y-5z= h, -2x +5y -9z = k is consistent, then:

A g+h+k=0

B 2g+h +k= 0

C g+h+2k=0

D g+2h+2k=0

atleast $|A_i| \neq 0$

$-\infty \quad |A| = 0$

$$A = \begin{vmatrix} 1 & -4 & 7 \\ 0 & 3 & -5 \\ -2 & 5 & -9 \end{vmatrix}$$

$= 1(-27+25) - 0(\ ) - 2(20-21)$

$= -2 + 2 = 0$

$6g + 3h + 3k = 0$

$2g + h + k = 0$

$A_1 \quad A_2 \quad A_3$

$$A_3 = \begin{vmatrix} 1 & -4 & g \\ 0 & 3 & h \\ -2 & 5 & k \end{vmatrix} = 0$$

$1(3k - 5h) - 2(-4h - 3g) = 0$

$3k + 3h + 6g = 0$

No
1
∞

$|A| \neq 0$ — 1

$|A| = 0$ — No at least $|A_i| \neq 0$; ∞ all $|A_i| =$

(1) nul
(2) rank

(iii)Gauss elimination Method for Non-homogeneous System

Let we have the non-homogeneous system $Ax=b$

where

m = No of Eq
n = No of var
r = Rank

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Now we write the augmented matrix of order $m \times (n+1)$

① $R_i \leftrightarrow R_j$ ✓
② $R_i \to kR_j$ ✓
③ $R_i \to R_i + kR_j$

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Now we can reduce this matrix in to row echelon form using elementary operations

Rank(A) = r

Row Rank
Column Rank
$\boxed{\text{Row Rank} = \text{Col. Rank}}$

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{ccccccc|c} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} & & b_1 \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2r} & \cdots & \bar{a}_{2n} & & \bar{b}_2 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \cdots & a^*_{rr} & \cdots & a^*_{rn} & & b^*_r \\ 0 & 0 & \cdots & 0 & \cdots & 0 & & b^*_{r+1} \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & & b^*_m \end{array}\right].$$

b^*_{r+2}

$(A|b) =$

Ex. Solve following system using gauss elimination

$$\text{(i)} \quad \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix},$$

$2x + y - z = 4$
$x - y + 2z = -2$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right)$$

$x - y + 2z = -2$

$-x + 2y - z = 2$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ -1 & 2 & -1 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\begin{array}{cccc} 1 & -1 & 2 & -2 \\ 0 & 3 & -5 & 8 \\ 0 & 1 & 1 & 0 \end{array}$$

$R_3 \to R_3 - \frac{1}{3}R_2$

$1 + \frac{5}{3}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 3 & -5 & 8 \\ 0 & 0 & \frac{8}{3} & -\frac{8}{3} \end{array}\right]$$

Rank(A) = 3

Rank(A|b) = 3

No of Variab = 3

$$\left[\begin{array}{ccc|} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 8/5 \end{array}\right]$$

$x - y + 2z = -2$

$3y - 5z = 8$

$\frac{8}{3} z = -\frac{8}{3}$

$z = -1$

$3y + 5 = 8$

$y = 1$

$x - 1 - 2 = -2$

$\boxed{x = 1}$

Note:

Rank(A) ≠ Rank(A|b) No Sol

1. Let $r < m$ and one or more elements $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are not zero. Then $rank(A) \neq rank(A|b)$ and the system of equation has no solution .
2. Let $m \geq n$ and $r = n$ (the number of columns in A) and $b^*_{\cdot+1}, b^*_{\cdot+2}, \ldots, b^*_{\imath}$ are all zero. In this case $rank(A) = rank(A|b) = n$ and the system of equations has unique solution.

Ran(A) = Ran(A|b) = n

3. Let $r < n$ and $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_m$ are all zero. In this case $x_1, x_2, \ldots, x_r$ can be determined in term of remaining (n-r)

$x_1, x_2, \dots, x_r$ can be determined in term of remaining (n-r) unknowns $x_{r+1}, x_{r+2}, \dots, x_n$.

$$\text{(ii)} \quad \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix},$$

$$\langle A|b\rangle = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 3 & 3 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 0 & 1 & 3 \\ 4 & -2 & 3 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & -1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & \boxed{-2} \end{array}\right] \quad b^*_{r+1}\ b^*_{r+2} \text{---} b^*_m$$

Rank(A) ≠ Rank(A|b)

Rank(A) = 2 Rank(A|b) = 3

No Sol

$$\text{(iii)} \quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}.$$

$$\langle A|b\rangle \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & -2 & 2 & 5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \\ \quad -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ \hline 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x - y + z = 1 \\ 3y - 3z = 0 \\ y = z \end{array}$$

$2 < 3$

$z = t$

0 0 0 0

$z = t$ $\quad 0 \quad 0 \quad 0 \quad 0$

$x = t \quad y = t \quad z = t$

$3y - 3z = 0$

$\boxed{y = 2}$

$x = 1$

(iv) Gauss-Jordan method

Ex. Using gauss-Jordan method solve the system of equations Ax=b, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -3 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right] \qquad R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 4 \\ 0 & +2 & 0 & 1 \end{array}\right] \qquad R_2 \to \frac{1}{3} R_2$$

$1 - \frac{5}{3}$

$0 - 2\left(-\frac{5}{3}\right)$

$1 - 2 \cdot \frac{4}{3}$

$3 - 8$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 2 & 0 & 1 \end{array}\right] \qquad R_1 \to R_1 + R_2, \quad R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & \frac{10}{3} & -\frac{5}{3} \end{array}\right] \qquad R_3 \to \frac{R_3}{\frac{10}{3}}$$

$\frac{4}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)$

$\frac{8-2}{6}$

$\frac{4}{3} + \frac{5}{3}\left(-\frac{1}{2}\right)$

$-\frac{5}{3} \cdot \frac{3}{10}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \qquad R_1 \to R_1 + \frac{2}{3} R_3, \quad R_2 \to R_2 + \frac{5}{3} R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x = 1, \; y = , \; z = $

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & \frac{1}{2} & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

~~$x = 1 \quad y = 1/2 \quad z = -1/2$~~

$$[\mathbf{A} \mid \mathbf{b}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{c}]$$

$R_3 \to R_3 - 3R_1$

$$[\mathbf{A} \mid \mathbf{b}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \cancel{\mathbf{c}}]$$

$x, y, z =$

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -3 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ R_3 - R_1 \end{array} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 4 \\ 0 & 2 & 0 & 1 \end{array}\right] R_2/3$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 2 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ R_3 - 2R_2 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 10/3 & -5/3 \end{array}\right] R_3/(10/3)$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \begin{array}{c} R_1 + 2R_3/3 \\ R_2 + 5R_3/3 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right].$$

$\mathbf{x} = [1 \quad 1/2 \quad -1/2]^T$.

Ex. Using Gauss-Jordan method find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$\begin{matrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{matrix}$$

$$[\mathbf{A} \mid \mathbf{I}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{A}^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & & & \\ 3 & -1 & & 1 & 0 & 0 \\ -1 & 3 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ 1 & -1 & -2 & & & \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 7 & -1 & 0 & 0 \end{array}\right] \qquad R_1 \to (-1)R_1$$

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 7 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to (-1) R_1$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} & & & -1 & 0 & 0 \\ & & & 3 & 1 & 0 \\ & & & -1 & 0 & 1 \end{array}\right]$$

$R_2 \to \frac{1}{2} R_2$

$-1 \quad -3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$R_3 \to R_3 - 2R_1$

?

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 0 \end{array}\right]$$

$R_3 \to \frac{1}{-5} R_3$

$(1, 0, 0)$
$(0, 1, 0)$ L.I
$(0\ 0\ 1)$

$A \cdot A^{-1} = I$

$|A| \neq 0$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

EIGENVALUES

$$\text{Let}\quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$AX = Y = \lambda X$

$AX = \lambda X$

Matr No

L.H.S = R.H.S

$AX = Y = \lambda X$

$AX = Y = \lambda X$

L.H.S = R.H.S

$$AX=Y \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where A is the matrix , X is the column vector and Y is also column vector.

Here column vector X is transformed into the column vector Y by means of the square matrix A.

Let X be a such vector which transforms into λX by means of the transformation (1). Suppose the linear transformation Y = AX transforms X into a scalar multiple of itself i.e. λX.

$$AX = Y = \lambda X$$
$$AX - \lambda IX = 0$$
$$(A - \lambda I)X = 0$$

$AX = \lambda X$

$[\ ] = [\ ]$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$

$A - \lambda I = 0$

$AB = 0 \quad A = 0, B = 0$

$(A - \lambda I)X = 0$

$\boxed{A - \lambda I = 0}$

$\det(A - \lambda I) = 0$

Ex. Find the eigenvalues and eigen vectors of the following matrices

det

(i) $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = A$

Note:

1. Characteristic Polynomial

2. Characteristic Equation

3. Characteristic Roots or Eigenvalues

(ii) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 3 & 5 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

Note1: Direct Characteristic equation for matrix A

Order 2: $\lambda^2 - trac(A)\lambda + \det(A) = 0$

Order 3:

$$\lambda^3 - trac(A)\lambda^2 + \big(Minor(a_{11}) + Minor(a_{22}) + Minor(a_{33})\big)\lambda - det(A) = 0$$

Note2: The eigenvalue of
(a) a symmetric/Hermitian matrix are real
(b) a skew-symmetric/skew-Hermitian matrix are zero or pure imaginary
(c)an orthogonal matrix are of magnitude 1 and are real or complex conjugate pairs
(d) an unitary matrix are of magnitude 1

Some Important Properties of Eigenvalues

(1) Any square matrix A and its transpose A' have the same eigenvalues.

(2) The sum of the eigenvalues of a matrix is equal to the trace of the matrix.

(3) The product of the eigenvalues of a matrix A is equal to the determinant of A.

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then the eigen values of

(i) kA are $k\lambda_1, k\,\lambda_2, \ldots, k\lambda_n$.
(ii)A^m are $\lambda_1^m,\ \lambda_2^m, \ldots, \lambda_n^m$.
(iii) A^{-1} $are \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

(5)$(A - kI)^{-1}$ has the eigenvalue $\frac{1}{\lambda - k}$.

(6) (A-kI) has the eigenvalue $\lambda - k$.

(7) For a real matrix A, if $\alpha + i\beta$ is an eigenvalue, then its conjugate $\alpha - i\beta$ is also an eigenvalue. When the matrix A is complex, this property does not hold

Theorem:(Cayley-Hamilton Theorem)

its conjugate $\alpha - i\beta$ is also an eigenvalue. When the matrix A is complex, this property does not hold.

Theorem:(Cayley-Hamilton Theorem)
Every square matrix A satisfies its own characteristic equation

Ex. Verify Cayley-Hamilton theorem for the following matrices. Also find the inverse of the matrix.

(i) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

CHARACTERISTIC VECTORS OR EIGEN VECTORS

A column vector X is transformed into column vector Y by means of a square matrix A.

Now we want to multiply the column vector X by a scalar quantity λ so that we can find the same transformed column vector Y. i.e., $AX = \lambda X$
X is known as eigenvector.

Show that the vector (1, 1, 2) is an eigen vector of the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$ *corresponding to the eigen value 2.*

Note: Corresponding to each characteristic root λ, we have a corresponding non-zero vector X which satisfies the equation $[A - \lambda I\,]\,X = 0$. The non-zero vector X is called characteristic vector or Eigen vector.

Ex. Find the eigenvalues and eigenvectors of the following matrices

(i) $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

PROPERTIES OF EIGEN VECTORS:

1. The eigen vector X of a matrix A is not unique.

2. If $\lambda_1, \lambda_2, \ldots., \lambda_n$ be distinct eigen values of an n × n matrix then corresponding eigen vectors $X_1, X_2, \ldots, X_n$ form a linearly independent set.

3. If two or more eigenvalues are equal it may or may not be possible to get linearly independent eigenvectors corresponding to the equal roots.

4. Two eigenvectors X_1 and X_2 are called orthogonal vectors if $X_1' X_2 = 0$.

5. Eigen vectors of a symmetric matrix corresponding to different eigenvalues are orthogonal.